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Complete description of forbidden subgraphs in the structural domination problem

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ABSTRACT

A class $\mathcal{D}$ of graphs is *concise* if it only contains connected graphs and is closed under taking connected induced subgraphs. This paper is concerned with concise classes of graphs. A graph G is $\mathcal{D}$-dominated if there exists a dominating subgraph $D \in \mathcal{D}$ in G. A connected graph G is minimal non-$\mathcal{D}$-dominated if it is not $\mathcal{D}$-dominated but all of its proper connected induced subgraphs are. We will give a complete description for the minimal non-$\mathcal{D}$-dominated graphs for a concise $\mathcal{D}$. The proof uses two stronger results.
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1. Definitions

For fundamental knowledge in domination, we refer to [16] and [23]. We consider finite, simple graphs only. As usual, by $V(G)$ and $E(G)$ we denote the vertex set and the edge set, respectively. K_n, P_n, C_n denotes the complete graph, the chordless path and cycle, respectively, on n vertices. $K_{p,q}$ denotes the complete bipartite graph with color classes of p and q vertices, respectively. A *hole* is a chordless cycle on at least 4 vertices. In this paper we shall deal with induced subgraphs. The graph G is H-*free* if it does not contain H as an induced subgraph. For a set $\mathcal{H}$ of graphs, the class of graphs which are H-free for every $H \in \mathcal{H}$, will be denoted by $\mathrm{Forb}(\mathcal{H})$. For a vertex v, $N(v)$ denotes the set of all neighbors of v, $N[v] := N(v) \cup \{v\}$. A vertex v is *universal* in G if $N[v] = V(G)$. A *star cut-set* in a graph $G = (V, E)$ is a vertex cut-set S such that for some vertex $v \in S$, $S \subseteq N[v]$. A set $D \subseteq V(G)$ is called *dominating* if for each $v \in V(G) - D$ there exists a neighbor $w \in D$ of v. The subgraph induced by a dominating set D is called a *dominating subgraph*. A set D is called k-*dominating* if for each $v \in V(G) - D$ there exists a $w \in D$ such that the distance of v and w is at most k. “Dominating” and “1-dominating” are equivalent. In the literature sometimes the name k-dominating is used for some different meaning.

Remark. In the further definitions a class of graphs $\mathcal{D}$ will play the main role. We emphasize that if the graph D is in $\mathcal{D}$ then all the graphs, isomorphic to D are in $\mathcal{D}$ but they are considered to be identical.

A class of connected graphs is *nontrivial* if it is nonempty and not the class of all connected graphs.

A class $\mathcal{D}$ of graphs is called *compact* if it is closed under taking connected induced subgraphs. A class is called *concise* if it is compact and it contains connected graphs only.

A graph G is *minimal not-in-*$\mathcal{D}$ if it is connected, is not in $\mathcal{D}$ but all of its proper connected induced subgraphs are. A graph G is $\mathcal{D}$-*dominated* if there exists a dominating connected induced subgraph $D \in \mathcal{D}$ in G. (Here we remark that in the case of a concise $\mathcal{D}$, the connectedness of D is automatically valid.) A graph G is called *hereditarily dominated by* $\mathcal{D}$ if each of its connected induced subgraphs is $\mathcal{D}$-dominated. The class $\mathrm{Dom}_k(\mathcal{D})$ contains the graphs G for which in all of their connected induced subgraphs H, H is k-dominated by some connected graph $D \in \mathcal{D}$. $\mathrm{Dom}(\mathcal{D}) = \mathrm{Dom}_1(\mathcal{D})$ will play

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the most important role. A *connected* graph G is *minimal non-$\mathcal{D}$-dominated* if it is not $\mathcal{D}$-dominated but all of its proper connected induced subgraphs are.

Let H be an induced subgraph of G and v a vertex of H. A vertex $v' \in V(G) - V(H)$ is a *private neighbor* of v if the edge $vv' \in E(G)$ is the only edge between v' and H. If H dominates G then, obviously, v has a private neighbor if and only if $H - v$ is not dominating in G.

Attaching (putting) a leaf to a given vertex v of H means to take a private neighbor v' of v, if any exists in G. If no supergraph G is given, we take v' as a new vertex.

The *leaf-graph* of a connected graph X can be defined independently of any other graph. It is the graph obtained from X by attaching a leaf to each of its *non-cutting* vertices. By definition, the leaves will be pairwise nonadjacent. The leaf-graph will be denoted by $F(X)$ and we say that X is the *core* of the graph $F(X)$. For example, $F(K_1) = K_2$.

Let us consider the graphs of the form $F(X)(X \neq K_1)$, K_1 itself and the holes. We shall call all of these graphs *fundamental graphs*.

2. Introduction

2.1. Problem specification

This paper is in some sense the final point of a research process that began twenty three years ago.

Among the first results of the common work with Zsolt Tuza, it was proved that a graph is P_5- and C_5-free if and only if all of its connected induced subgraphs have a dominating clique [2]. Cozzens and Kelleher [11] also obtained this result independently, and later several generalizations were found, as will be mentioned below.

General question

Given a class $\mathcal{D}$ of connected graphs, characterize the class of those graphs in which every connected induced subgraph contains a dominating induced subgraph isomorphic to some $D \in \mathcal{D}$.

This is what we call structural domination. The result mentioned above can be formulated in the following language. Let us denote the class of all cliques by $\mathcal{D}_1$. The class to be characterized is the class of P_5- and C_5-free graphs. Then

$$\mathrm{Dom}(\mathcal{D}_1) = \mathrm{Forb}(P_5, C_5).$$

The first result of this type can be found in Wolk's paper [22]. It states that a graph is hereditarily dominated by a K_1 if and only if it is P_4- and C_4-free:

$$\mathrm{Dom}(\{K_1\}) = \mathrm{Forb}(P_4, C_4).$$

Hereditarily dominated graphs have been characterized for some further families of graphs, too, e.g. for $\mathcal{D} = \{G : \mathrm{diam}(G) \leq t\}$ for every given $t \geq 2$ in [5]. Let us remark that this class $\mathcal{D}$ is noncompact.

In this work we mostly restrict ourselves to the following:

The central problem

Given a concise and nontrivial class $\mathcal{D}$ of graphs, determine the collection of minimal non-$\mathcal{D}$-dominated graphs.

This problem was treated for example in [6]. The counterpart of the *Main Theorem* of Section 3 was proved there (the so-called Cut-point Lemma).

2.2. History of the problem

Some results related to the central problem are the following. El Zahar and Erdős [13] proved that in a $2K_2$-free connected graph a dominating P_3 or clique always exists. In [2] we generalized this result, and in [6] we gave a complete characterization of graphs having this domination property (in the structural sense). Further domination properties were proved for $2K_2$-free and P_5-free graphs, respectively, in [9] and in [2]. Let us note that most of these results are the generalizations of the work of the 'ancestor', Seinsche [19]. Some generalizations to k-domination have also been considered [3–6].

Penrice [18] raised the question how it can be ensured that a particular connected graph has a connected dominating subgraph so that the clique covering number does not exceed a given bound. Some of our questions in [3] concerning the set of vertices that i-dominate the whole graph were answered by Dong [12]. These sets were discussed in [14] as well (see above); furthermore, Favaron and Fouquet [15] gave counterexamples for some of our conjectures in the subject, while they proved a weaker version of our conjecture. Liu and Zhou [17] characterized the graphs hereditarily dominated by the family of complete bipartite graphs within the class of K_3-free graphs. In [1] a generalization of [17] was proved. The minimal non-dominated graphs were also found for the class of complete bipartite graphs and the same question for the class of dominating connected bipartite graphs was discussed there in general. Zverovich [21] has shown that in a connected P_5- and C_5-free graph, the connected domination number γ_c and the traditional domination number γ are always equal, and [7] has proved that even a dominating clique of size γ exists in such graphs. [8] dealt with the existence of dominating paths. The appropriate minimal non-dominated graphs were also described, both for unbounded and bounded path lengths. As we shall see, dominating chordless paths play a central role in the general problem, as well. Here we remark that the early investigations have shown already the connection of the problem with star-cutsets defined by Chvátal [10] when collecting the properties of minimal imperfect graphs.

2.3. On the main result of the paper

The non-2-connected case of the Central Problem was solved in [6]:

Cut-point Lemma (*[6]*). *Let $\mathcal{D}$ be a concise and nontrivial class. A graph G with at least one cut-point is minimal non-$\mathcal{D}$-dominated if and only if it is isomorphic to a leaf-graph $F(L)$, where $L \neq K_1$ is a graph minimal not-in-$\mathcal{D}$.*

Of course, sometimes even the problem of finding all the graphs minimal not-in-$\mathcal{D}$ can be difficult, e.g. for $\mathcal{D} = \{$the connected perfect graphs$\}$. This was an open problem for 40 years. Anyway, the *Cut-point Lemma* and the *Main Theorem* (see below, in Section 3) reduce the problem of minimal non-$\mathcal{D}$-dominated graphs to the problem of graphs minimal not-in-$\mathcal{D}$. For years, it seemed that for several $\mathcal{D}$'s, some 2-connected minimal non-$\mathcal{D}$-dominated graph can be found, besides the cycles. Finally, the proof for the nonexistence of this type of graphs was established. The reader can find it in this paper ("*Main Theorem*"), deduced from stronger results (*King Theorem*, Theorem 1). Zsolt Tuza [20] has made an independent proof for the *Main Theorem*. Thus, in the compact case for 2-connected graphs the problem is much more difficult than for non-2-connected graphs, while, for 2-connected graphs, compact and noncompact case, the proofs are equivalent.

3. The results

We have a graph class $\mathcal{D}$. It can be of two types:
Type 1. All the chordless paths are the elements of $\mathcal{D}$.
Type 2. There exists an integer $t \geq 3$ such that t is the minimal subscript with $P_t \notin \mathcal{D}$.

Main Theorem. *Let $\mathcal{D}$ be a concise and nontrivial class of graphs. If $\mathcal{D}$ is of Type* 1 *then there is no* 2*-connected minimal non-$\mathcal{D}$-dominated graph. If $\mathcal{D}$ is of Type* 2 *then the only* 2*-connected minimal non-$\mathcal{D}$-dominated graph is the cycle C_{t+2}. The non-* 2*-connected case is described by the Cut-point Lemma (see Introduction).*

Theorem for the noncompact case. *Let $\mathcal{D}$ be any nontrivial class of graphs, consisting of connected graphs and let G be a* 2*-connected minimal non-$\mathcal{D}$-dominated graph. Then G is a cycle (if any such G exists).*

We shall state two stronger theorems. In the next section we shall prove them and we shall deduce the *Main Theorem* from them. We introduce the notion of *king* and the *King Theorem* as follows:

Definition 1. Given a connected graph G, a *proper* connected induced subgraph K is a *king* of G if every connected dominating subgraph of K is dominating in the whole G as well. (*We remark that a graph may have several kings.*)

We need the following statement.

Proposition 1. *Every connected dominating subgraph of $F(X)$ contains all the vertices of X, except if $X = K_1$.*

Proof. Such a graph must contain all the non-cutting vertices of X and, by a well-known fact, this implies that it contains the whole X. □

A simple but important fact:

Proposition 2. *The fundamental graphs have no kings.*

Sketch of proof. Most of the proof can be managed by Proposition 1. □

We are now in the position to state the

King Theorem. *If a connected graph is not fundamental then it contains a king.*

Though we shall not use the strengthening below in the proof of the *Main Theorem*, we state it:

Strong King Theorem. *A connected graph has a king if and only if it is not fundamental. In this case, it has even a king that is a fundamental graph.*

Now we give the axioms for stating Theorem 1.
Let $\mathcal{X}$ consist of some connected induced subgraphs of a fixed connected graph G.
(a) $\mathcal{X} \neq \emptyset$
(b) There are no chordless paths in $\mathcal{X}$.
(c) Let $H \in \mathcal{X}$ and let v be a non-cutting vertex in H. If $H - v \notin \mathcal{X}$ then a leaf can be put on H in v in such a way that the graph obtained so is in $\mathcal{X}$ again.
Furthermore, we define the following property:
(1) There exists some graph X such that for $F = F(X)$, $F, X \in \mathcal{X}$ at the same time and X is the core of F. (We mean that X is not only isomorphic to the core but the core itself.)

Remark. The members of $\mathcal{X}$ are managed in another manner than those of $\mathcal{D}$ above. The different induced subgraphs of G are considered to be different, even if they are isomorphic.

Theorem 1. *If the set $\mathcal{X}$ of graphs satisfies axioms* (a), (b), (c) *then G has Property* (1).

4. The proofs

Now we introduce a notion that is necessary for the proof of Theorem 1.

Definition 2. P is a path graph of the connected graph M if it can be constructed from M by putting vertex-disjoint chordless paths on some of its non-cutting vertices, at most one path on one vertex. By definition, every connected graph is the path graph of itself.

Remark. The path graph of a graph is rarely a path.

Notation. For a fixed connected graph M, the set of the path graphs of M is denoted by $\mathcal{P}(M)$.

Definition 3. The connected graph N is a non-path-graph if it is not the path graph of any smaller graph.

Claim 1. *Every connected graph P, different from a chordless path, can be uniquely expressed as the path graph of a non-path-graph. (The latter can be called the basis of P.)*

Explanation. If G has no leaves, it is a non-path-graph and it is the basis of itself. Otherwise, starting from a leaf, going through vertices of degree 2, we find a vertex of larger degree. If this is a non-cutting vertex of the remaining graph, we may omit the path constructed. If we get a cutting vertex c then there are two cases. If the path contains the leaf only then we did not find a path to attach. Otherwise, the non-cutting vertex will be the neighbor of c on the path and we have to attach the remaining path on it. $\square$

Remark. A chordless path on more than one vertex cannot be uniquely expressed since both endpoints yield a basis graph from which the graph can be built.

Proof of Theorem 1. Let $\mathcal{X}$ be a counterexample for Theorem 1. Furthermore, let $P \in \mathcal{X}$ and let M be the basis of P (M is uniquely determined, because of (b) and Claim 1).

Notation. First, let us introduce for a connected graph M, the notation $\kappa(M)$ for the number of non-cutting vertices in M.

Let M have the following property, called *basis minimality condition*.
– Among the non-path-graphs M^* for which there exists some $P \in \mathcal{P}(M^*) \cap \mathcal{X}$, M has minimum κ.
By Axiom (b), P is not a chordless path and thus, neither is M. M has at least 3 non-cutting vertices, we denote them by $y_1, y_2, \ldots, y_k$, where $k = \kappa(M)$. Let $U_i = U_i(P)$ be the path hanging on y_i. (U_i may consist of y_i only.)

Notation. For any path W, let $|W|$ denote the number of edges in W.

Now we shall change P in the following way:
$s_1 := \min\{|U_1(P)| : P \in \mathcal{P}(M) \cap \mathcal{X}\}$ Remark: s_1 may be zero.
We can choose P so that $|U_1(P)| = s_1$ be valid. This choice is not unique but possible.
We define recursively $s_i := \min\{|U_i(P)| : P \in \mathcal{P}(M) \cap \mathcal{X}$ with $|U_j(P)| = s_j$ for $j = 1, \ldots, i-1\}$, $i = 2, \ldots, k$.
Remark: s_i also may be zero.
We continue to transform P:
Let the endpoint of U_i be v_i ($v_i = y_i$ is allowed.) First, we omit v_k from P. We state that the remaining graph is not in $\mathcal{X}$. The reason is the minimality of U_k if $v_k \neq y_k$.
Otherwise we need a statement for the situation when we omit a special vertex from a path graph:

Lemma 1. *Let Q be a connected graph with basis N and let y be a non-cutting vertex of N so that there is no path attached to N in y. If Q–y is not a path then, calling its basis N', $\kappa(N') < \kappa(N)$.*

Proof. Let us consider an arbitrary non-cutting vertex u of N'. We distinguish three types of such vertices:
Type 1. u is non-cutting in N as well.
Type 2. u is a cut-point in N.
Type 3. u is not a vertex of N.
We will show that Type 2 may be supposed not to occur and if u is of Type 3, there exists a non-cutting vertex $v = v(u)$ of N which is cutting in N'; furthermore, for different vertices u, the corresponding vertices $v(u)$ are different and differ from the vertices of Type 1. The vertex y is non-cutting in N and it is not in $V(N')$, consequently these facts will imply $\kappa(N') < \kappa(N)$. So, this will be sufficient for our proof.

Claim 2. *We may suppose that Type 2 does not occur.*

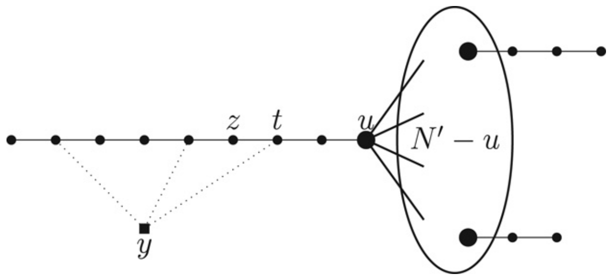

Fig. 1. A vertex of Type 2.

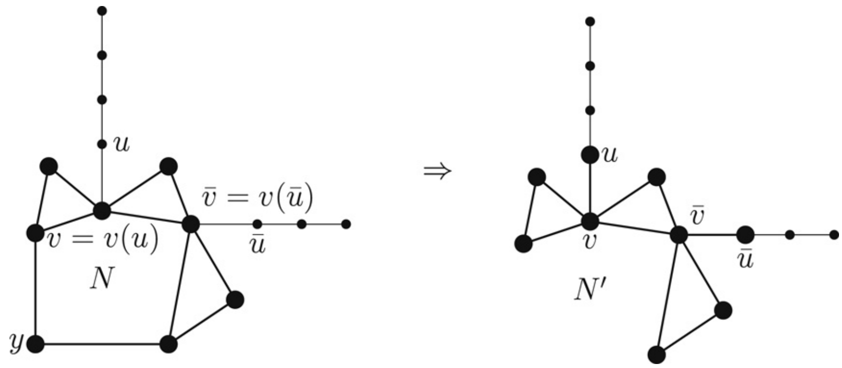

Fig. 2. Change in the basis.

Proof. If u is cutting in N then it is also cutting in Q which is a path graph of N. We know that u is non-cutting in N', consequently, its path graph Q–y has the following structure: a connected graph N'–u and some paths attached to it, u, and (perhaps) a path attached to u. How can we add y to this graph so that u becomes a cut-vertex? If a path is attached to u, all the neighbors of y lie on this path L. (See Fig. 1.) The *Lemma* is true in this case as well. The reason: Let t be the neighbor of y nearest to u and let z be the neighbor of t on L, far from u. Both y and z are extra non-cutting vertices of N, so $\kappa(N) > \kappa(N')$. The only exception is when y has exactly one neighbor on L: the endpoint. In this case y is not in the basis since it is on a path, attached to u. (Remark that Type 2 occurs here and this is the only occasion.) In the other case (no attached path), y has one neighbor in the whole graph Q: u. But this is impossible, since y is not in the basis, similarly as above. Claim 2 is proved. □

Let us consider Type 3. It can be easily seen that any vertex $u \in V(N') - V(N)$ is the first vertex on a path attached to N in a vertex v which is a non-cutting vertex of N and $\{y, v\}$ is a cutset.

Thus, v is convenient for $v(u)$ since it is a non-cutting vertex of N and a cut-point of N'. And of course, $v(u_1) \neq v(u_2)$ for $u_1 \neq u_2$. Furthermore, if $\tilde{u}$ is of Type 1 and u of Type 3 then $\tilde{u} = v(u)$ is impossible since in N', $\tilde{u}$ is a non-cutting vertex, while $v(u)$ is a cutpoint. (See Fig. 2.) We have shown all the facts, necessary for the proof of Lemma 1. □

Thus we are also done when $v_k = y_k$. Let us apply the Lemma for $Q = P$, $N = M$ and $y = y_k$. We obtain that $P - v_k \notin \mathcal{X}$. If $P - v_k$ is a path then this is the consequence of Axiom (b). Otherwise it comes from the *basis minimality condition*.

Using Axiom (c), some graph R_1, obtained by putting a leaf on v_k, is in $\mathcal{X}$. Now we begin to decrease i. In the general step, we omit v_i from the graph R_{k-i}. If necessary, we may apply Lemma 1, thus the remaining graph will not be in $\mathcal{X}$, similarly to the above case. So, by (c), we may put a leaf on v_i and we get a graph R_{k+1-i}, being in $\mathcal{X}$.

Finally, we obtain a graph $R_k \in \mathcal{X}$ which is the leaf graph $F(P)$ of P and this proves Property (1). Theorem 1 is proved. □

Proof for the King Theorem I. First we prove an important auxiliary statement in the subject.

Lemma 2. *In a counterexample for the King Theorem, there is no dominating chordless path.*

Definition 4. Given a hole C, a *partial path* of C is a path, obtained from C, by deleting two adjacent vertices of C.

Proof of Lemma 2. If we assume that there exist some dominating chordless paths in G, we may also assume that we have a minimal one, denoted by U. The case of a "one-vertex path" is trivial. We suppose a path of several vertices.

The following simple statement will be useful:

Claim 3. *Let U be a minimal dominating chordless path in G. Let the endpoints a and b of U be different. Let A and B be the sets of private neighbors of a and b with respect to U. None of A and B are empty and for any $a' \in A$ and $b' \in B$, $a'b' \in E(G)$.*

Sketch of proof. If the middle part of a chordless path is dominating then it is a king and there is no king in G. □

Furthermore, there exists a hole C for which U is a partial path. (C can be made by adding some $a' \in A$ and $b' \in B$ to U).
We shall prove now the central claim of Lemma 2:

Claim 4. *Let C be a hole in G and let U be a partial path of C which is minimal dominating in G. Furthermore, let W be a partial path of C which is the "neighbor" of U. Then W is also minimal dominating in G.*

Proof. Let W be the path obtained from C, by deleting a and a' from C. It is enough to prove that W is minimal dominating in G.

Suppose first there exists a vertex w not dominated by W. This vertex is dominated by U but not dominated by any vertex of the set $U - a$. Consequently, it is dominated by a. Also $w \in A$ is valid. But this is a contradiction since, by Claim 3, $wb' \in E(G)$.

The minimality of W is easy to prove. Deleting any endpoint of W, the rest does not dominate even C. □

Using Claim 4, we obtain that every partial path of C is dominating. So, obviously, C is a king in G. Lemma 2 is proved. □

II. We prove that Theorem 1 implies the King Theorem. Let G be any counterexample for the *King Theorem*. Let $\mathcal{X}$ be the set of those connected induced subgraphs of G which dominate it.

By Lemma 2, $\mathcal{X}$ satisfies Axiom (b). Trivially, it satisfies Axioms (a) and (c) too. By Theorem 1, $\mathcal{X}$ has Property (1). So G contains an induced subgraph F isomorphic to $F(X)$ for some connected induced subgraph X and both F and X are in $\mathcal{X}$, so, X is dominating in G. Then, using Proposition 1, we obtain that F is a king in G.

Thus the *King Theorem* is proved. □

Proof for the Main Theorem (*And the Theorem for the Noncompact Case*). We shall prove that the *King Theorem* implies both theorems.

Let us take a minimal non-$\mathcal{D}$-dominated graph G which is 2-connected. It can be supposed that G is not a cycle. Thus, G is not fundamental, and the *King Theorem* can be applied and G has some king K.

K is a proper induced subgraph of G and, since G is minimal non-$\mathcal{D}$-dominated, K is dominated by some of its induced connected subgraphs $D \in \mathcal{D}$. But, by the properties of a king, D dominates G as well—this contradicts the fact that G is not $\mathcal{D}$-dominated. We have deduced both theorems from the *King Theorem*. □

Proof of the Strong King Theorem (*As a Remark*). We have chosen this proof to be the last because this statement is not in the main stream of the paper.

The proof will show that King Theorem is a self-strengthening statement.

Let us consider a non-fundamental graph G. By the King Theorem, it has some kings. Let us pick a minimal one, that is, a king K_1 which does not contain any proper induced subgraph which is a king of G. We state that K_1 is a fundamental graph. By way of contradiction, suppose K_1 is non-fundamental. Then repeatedly by the King Theorem, K_1 has a king K_2. It can be easily seen that "king of a king is a king". Thus, K_2 is a king of G as well, contradicting the minimality of K_1. □

5. Examples

Example 1/a. For a graph satisfying the conditions of the *King Theorem* but having no universal vertex and having no induced connected subgraph X with the following two properties:
– X is dominating in G.
– X can be extended to $F(X)$ within G.

The line graph of $K_{3,3}$ is an example. Of course, it does have holes which are kings in the graph.

Example 1/b. For the reverse case when holes are missing among the kings. This is much more frequent, e.g. any triangulated graph (i.e., graphs without holes) with vertex degrees at least 2.

Explanation. Such a graph G is not fundamental since it is not of the form $F(X)$, because of the degree condition, neither a hole, nor a K_1. The King Theorem can be applied for G and the kings will not form holes.

Example 2. For a connected graph G and a nonempty set $\mathcal{X}$ of its connected induced subgraphs such that $\mathcal{X}$ satisfies Axioms (a), (b) and (c) but $\mathcal{X}$ is not the set of dominating subgraphs in G. (Of course, $\mathcal{X}$ does have Property (1).) The vertex set $V(G)$ is $\{a, b, c, d, e, f, g\}$, $E(G)$ is $\{ab, ac, ad, be, cf, dg, ef, fg, ge\}$. $\mathcal{X}$ consists of the subgraphs induced by the vertex sets $\{e, f, g\} \cup S$ where S is an arbitrary subset of $\{b, c, d\}$. The subgraph induced by $\{e, f, g\}$ is in $\mathcal{X}$ and is non-dominating in G.

Remark. Probably numerous similar sets of graphs $\mathcal{X}$ exist. We plan to investigate them and to apply Theorem 1 for them in future work.

Example 3. For a king that contains a dominating subgraph which is not dominating in the whole graph:

The graph G is a bull on vertex set $V = \{a, b, c, d, e\}$ and edge set $\{ab, ad, bd, bc, de\}$. The king K is $\{b, c, d, e\}$. The subgraph induced by the set $\{c, e\}$ is dominating in K but not in G.

Of course, every subgraph of this kind is disconnected.

Example 4. For graphs with kings which are not fundamental graphs. The following is true:

Proposition 3. *Given any connected graph K, there are infinitely many pairwise non-isomorphic graphs G, having K as a king.*

Proof. Let us fix a copy of K and let $\mathcal{H}$ be the hypergraph having the vertex sets of the minimal connected dominating subgraphs of K as hyper-edges. If $V(G)$ contains $V(K)$ then K is a king in G if for all the vertices $x \in V(G) - V(K)$, $N(x) \cap V(K)$ intersects every hyper-edge in $\mathcal{H}$. The structure of the subgraph induced by $V(G) - V(K)$ does not play any role here. $\square$

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